On the effect of the domain geometry on the existence of sign changing solutions to elliptic problems with critical and supercritical growth

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Abstract

This paper deals with the existence of sign changing solutions of problem

\[
\begin{aligned}
-\Delta u &= |u|^{p-1}u + \varepsilon w(x)|u|^{q-1}u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

where \( \Omega \) is a bounded regular domain in \( \mathbb{R}^N \), \( N \geq 4, \varepsilon > 0, p = \frac{N+2}{N-2} \), \( q \geq 1, q \neq p \) and \( w \in C^1(\Omega) \).

Key words: Critical Sobolev exponent, sign changing solutions.

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1 Introduction and main results

In this paper we are concerned with sign changing solutions to the problem

\[
\begin{aligned}
-\Delta u &= |u|^{p-1}u + \varepsilon w(x)|u|^{q-1}u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]
where $\Omega$ is a bounded regular domain in $\mathbb{R}^N$, $N \geq 4$, $\varepsilon > 0$, $p = \frac{N+2}{N-2}$, $w \in C^1(\overline{\Omega})$ and $q \geq 1$, $q \neq p$.

We are interested in the existence of sign changing solutions which blow-up positively and blow-up negatively at different points of $\Omega$ as the parameter $\varepsilon$ goes to 0 in the sense of the following definition.

**Definition 1.1** Let $u_\varepsilon$ be a family of solutions for (1.1). We say that $u_\varepsilon$ blow-up positively at $k_1$ different points $\xi_1, \ldots, \xi_{k_1}$ in $\Omega$ and blow-up negatively at $k_2$ different points $\xi_{k_1+1}, \ldots, \xi_k$ ($k := k_1 + k_2$) in $\Omega$ if there exist $k$ rates of concentration $\mu_1, \ldots, \mu_k$ and $k$ points $x_1, \ldots, x_k \in \Omega$ with $\lim_{\varepsilon \to 0} \mu_i = 0$ and $\lim_{\varepsilon \to 0} x_i = \xi_i$ such that

$$u_\varepsilon - \left( \sum_{i=1}^{k_1} P_{\Omega} U_{\mu_i, x_i} - \sum_{i=k_1+1}^{k} P_{\Omega} U_{\mu_i, x_i} \right) \to 0 \quad \text{in} \quad H^1_0(\Omega) \quad \text{as} \quad \varepsilon \to 0.$$ 

Here (see [3], [8] and [22])

$$U_{\mu,y}(x) = C_N \frac{\mu^{N-2}}{(\mu^2 + |x-y|^2)^{\frac{N-2}{2}}}, \quad x \in \mathbb{R}^N, \quad y \in \mathbb{R}^N, \quad \mu > 0,$$

with $C_N = [N(N-2)]^{(N-2)/4}$, are all the positive solutions of the problem

$$-\Delta U = U^{\frac{N+2}{N-2}} \quad \text{in} \quad \mathbb{R}^N$$

and $P_{\Omega} U_{\mu,y}$ is the projection onto $H^1_0(\Omega)$ of $U_{\mu,y}$, i.e.

$$\begin{cases} \Delta P_{\Omega} U_{\mu,y} = \Delta U_{\mu,y} & \text{in} \ \Omega, \\ P_{\Omega} U_{\mu,y} = 0 & \text{on} \ \partial \Omega. \end{cases}$$

In this paper we consider both the subcritical and the supercritical case and we treat both the autonomous and the non-autonomous case. Our first result concerns the existence and the profile of sign changing solutions to problem (1.1), with at least two nodal domains.

**Theorem 1.2** Let $q > p$ and $\max_{\overline{\Omega}} w < 0$. Then for $\varepsilon$ small enough problem (1.1) has at least a pair of sign changing solutions, which blow-up positively and negatively at two different points of $\Omega$.

**Theorem 1.3** Let $\max_{\overline{\Omega}} \{1, p-1\} \leq q < p$ and $\min_{\overline{\Omega}} w > 0$. Then for $\varepsilon$ small enough problem (1.1) has at least a pair of sign changing solutions, which blow-up positively and negatively at two different points of $\Omega$. 
In the next result for any $k$ fixed integer we construct a domain like a “dumb-bell with thin handles” for which problem (1.1) has many sign changing solutions with at least $k$ nodal domains.

**Theorem 1.4** Assume either $q > p$ and $\max_{\Omega} w < 0$ or $q \in [1, p)$ if $N \geq 5$, $q \in (1, p)$ if $N = 4$ and $\min_{\Omega} w > 0$. Let $k \geq 2$ be a fixed integer. There exists a contractible domain $\Omega$ such that for $\varepsilon$ small enough problem (1.1) has at least $2k-1$ pairs of sign changing solutions, which blow-up positively at $h$ points and blow-up negatively at $k-h$ points.

Finally we consider the symmetric case and we give a different proof of the result found by Fortunato and Jannelli in [13] (we also extend it to the supercritical case), by describing the profile of the solutions to (1.1) as the parameter $\varepsilon$ goes to zero.

**Theorem 1.5** Assume $\Omega$ is a ball. Assume either $q > p$ and $w \equiv -1$, or $q \in [1, p)$ if $N \geq 5$, $q \in (1, p)$ if $N = 4$ and $w \equiv 1$. Let $h$ be a fixed integer. Then for $\varepsilon$ small enough problem (1.1) has at least a pair of sign changing solutions, which blow-up positively at $h$ points and blow-up negatively at $h$ points.

Let us recall some known results. We introduce some notation.

Let us denote by $G$ the Green’s function of the negative laplacian on $\Omega$ and by $H$ its regular part, choosen in such a way that

$$H(x, y) = \frac{\alpha_N}{|x-y|^{N-2}} - G(x, y), \quad \forall (x, y) \in \Omega^2,$$

where $\alpha_N = \left[ (N-2)\text{meas}(S^{N-1}) \right]^{-1}$ and $S^{N-1}$ is the $(N-1)$-dimensional unit sphere. Let $\tau(x) := H(x, x)$ be the Robin function of $\Omega$ at the point $x \in \Omega$.

First of all we consider the autonomous case, i.e. $w(x) = \omega \in \mathbb{R}$ for any $x \in \Omega$. It is well known that Pohozaev’s identity allows us to prove that if the domain $\Omega$ is starshaped and either $\omega < 0$ and $q \in [1, p)$ or $\omega > 0$ and $q > p$, then for any $\varepsilon \geq 0$ problem (1.1) has no nontrivial solutions.

Let us recall some known results about the existence of positive solutions to (1.1). In the subcritical case, i.e. $q \in [1, p)$, Brezis and Nirenberg in [7] proved that if $N \geq 4$ and $\omega > 0$, for $\varepsilon$ small enough problem (1.1) has a family of positive solutions, which blow-up at a critical point of the Robin’s function (see [14], [20] and [21]). Moreover if $q = 1$, in [21] and [19], it was
proved that any stable critical point $\xi_0$ of the Robin’s function generates a family of positive solutions which blow-up at $\xi_0$. In the supercritical case, i.e. $q > p$, Merle and Peletier in [17] proved that if $N \geq 3$ and $\omega < 0$, for $\varepsilon$ small enough problem (1.1) has a family of positive solutions, which blow-up at a critical point of the Robin’s function. Moreover in [18], it was proved that any stable critical point $\xi_0$ of the Robin’s function generates a family of positive solutions to (1.1) which blow-up at $\xi_0$.

Let us recall some known results about the existence of sign changing solutions to (1.1) which are obtained when $q = 1$.

The first result is due to Cerami, Solimini and Struwe, who showed in [10] the existence of a pair of least energy sign changing solutions if $N \geq 6$ and $\varepsilon$ is small enough. Some multiplicity result for sign changing solutions are also known. Fortunato and Jannelli in [13] consider a domain with cylindrical or rotational symmetry and they prove that if $N \geq 4$ and $\varepsilon > 0$ problem (1.1) has infinitely many sign changing solutions which exhibit some symmetries. In particular if, for example, $\Omega$ is a ball those solutions are not radial. If $\Omega$ is a ball other results are known. If $N \geq 7$ in [10] it was proved the existence of infinitely many radial solutions which change sign provided $\varepsilon$ is small enough. On the other hand, if $N = 4, 5, 6$ in [1] and [2] it was proved that problem (1.1) has no radial solutions which change sign if $\varepsilon$ is small enough. The existence of sign changing radial solutions to (1.1) when $\Omega$ is a ball and $q \in (p - 1, p)$ was established by Jones in [16]. He also proved that if $q \in (1, p - 1)$ and $\varepsilon$ is small enough, problem (1.1) has no sign changing radial solutions.

The existence of infinitely many solutions to (1.1) for a general domain $\Omega$ and for any $\varepsilon > 0$ was established by Devillanova and Solimini in [11] when $N \geq 7$. For low dimension, namely $N = 4, 5, 6$, in [12] the authors proved the existence of at least $N + 1$ pairs of solutions to (1.1) provided $\varepsilon$ is small enough. We would like also to quote the paper [5] by Bartsch, where the author proves the existence of infinitely many sign changing solutions to the subcritical problem $-\Delta u = |u|^{p-1}u + \lambda u$ in $\Omega$, $u = 0$ on $\partial \Omega$, with $1 < p < \frac{N+2}{N-2}$.

In [9] Castro and Clapp consider a domain $\Omega$ which is invariant under an orthogonal involution (for example, $\Omega$ is symmetric with respect to the origin or it has a cylindrical or rotational symmetry) and they prove the existence of one pair of solutions of (1.1) which change sign exactly once, provided $N \geq 4$ and $\varepsilon$ is small enough. Moreover they describe the profile of the solutions, by showing that the solutions blow-up positively and negatively at two different points in $\Omega$ as $\varepsilon$ goes to 0, according to Definition 1.1.

As far as it concerns the non-autonomous case, some results about the ex-
istence of positive solutions are obtained in [18]. More precisely the authors proved that if $\xi_0$ is a stable critical point of the function $\psi(\xi) := w(\xi)\tau(\xi) \frac{\eta}{2}$ and either $q \in [1, p)$, $w(\xi_0) > 0$ or $q > p$, $w(\xi_0) < 0$, then for $\varepsilon$ small enough there exists a family of positive solutions to (1.1) which blow-up at $\xi_0$.

It seems that there is not any results about the existence of sign changing solutions in the supercritical case or in the non-autonomous case.

The proof of our results is based on a Lyapunov-Schmidt procedure developed in [18], which enables us to treat both the supercritical case and the subcritical case.

The paper is organized as follows: in Section 2 we recall some known results and we reduce the problem to a finite dimensional one, in Section 3 we study the reduced problem and we prove Theorem 1.2, Theorem 1.3 and Theorem 1.4, in Section 4 we consider the symmetric case and we prove Theorem 1.5, in Appendix there are some useful estimates.

2 Setting of the problem and the finite dimensional reduction

Here we recall some results obtained in [18].

In the following we assume $\alpha > 0$ and set $\Omega_\varepsilon = \Omega/\varepsilon^\alpha$. An easy computation shows that, if $u(x)$ solves problem (1.1), then $v(y) = \varepsilon^{\frac{\alpha N}{2}} u(\varepsilon y)$ solves

$$\begin{cases}
-\Delta v = v^p + \varepsilon^\eta w(\varepsilon y)v^q & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega
\end{cases} \quad (2.1)$$

where

$$\eta = 1 + \alpha \frac{N + 2}{2} - q\alpha \frac{N - 2}{2}. \quad (2.2)$$

The parameter $\alpha$ will be chosen later in a suitable way.

Let $H_0^1(\Omega_\varepsilon)$ be the Hilbert space equipped with the usual inner product

$$\langle u, v \rangle = \int_{\Omega_\varepsilon} \nabla u \nabla v, \text{ which induces the norm } \|u\|_{H_0^1(\Omega_\varepsilon)} = \left( \int_{\Omega_\varepsilon} |\nabla u|^2 \right)^{1/2}.$$  

Moreover, if $r \in [1, +\infty)$ and $u \in L^r(\Omega_\varepsilon)$, we will set $\|u\|_r = \left( \int_{\Omega_\varepsilon} |u|^r \right)^{1/r}$.  

It will be useful to rewrite problem (2.1) in a different setting. Let us then introduce the following operator.
Definition 2.1 Let $i^*_\varepsilon : L^{2N/2N-2}(\Omega_\varepsilon) \to H^1_0(\Omega_\varepsilon)$ be the adjoint operator of the immersion $i_\varepsilon : H^1_0(\Omega_\varepsilon) \hookrightarrow L^{2N/2N-2}(\Omega_\varepsilon)$, i.e.

$$i^*_\varepsilon(u) = v \iff (v, \varphi) = \int_{\Omega_\varepsilon} u(x) \varphi(x) \, dx \quad \forall \varphi \in H^1_0(\Omega_\varepsilon).$$

Remark 2.2 The operator $i^*_\varepsilon : L^{2N/2N-2}(\Omega_\varepsilon) \to H^1_0(\Omega_\varepsilon)$ is continuous uniformly with respect to $\varepsilon$, as

$$\|i^*_\varepsilon(u)\|_{H^1_0(\Omega_\varepsilon)} \leq S^{-\frac{1}{2}} \|u\|_{L^{2N/2N-2}} \quad \forall u \in L^{2N/2N-2}(\Omega_\varepsilon), \quad \forall \varepsilon > 0,$$

(S is the best constant for the Sobolev embedding).

Arguing as in [18], we can prove the following result.

Lemma 2.3 Let $s > \frac{N}{N-2}$. If $u \in L^{Ns/2}(\Omega_\varepsilon) \cap L^{2N/2N-2}(\Omega_\varepsilon)$ then $i^*_\varepsilon(u) \in L^s(\Omega_\varepsilon)$ and

$$\|i^*_\varepsilon(u)\|_s \leq C(\Omega) \|u\|_{L^{Ns/2}} \quad \forall \varepsilon > 0.$$

Let

$$X = H^1_0(\Omega_\varepsilon) \cap L^s(\Omega_\varepsilon)$$

where $s = \frac{2N}{N-2}$ if $q \leq p$ or $s$ is large enough if $q > p$. For the sake of simplicity we will choose $s = Nq$ if $q > p$.

$X$ is a Banach space equipped with the norm

$$\|u\| = \|u\|_{H^1_0(\Omega_\varepsilon)} + \|u\|_s.$$

We point out that if $q \leq p$ the space $X$ coincides with $H^1_0(\Omega_\varepsilon)$.

By means of the definition of the operator $i^*_\varepsilon$, problem (2.1) turns out to be equivalent to

$$\begin{cases} u = i^*_\varepsilon[f(u) + \varepsilon^q w(\varepsilon^\alpha y) g(u)] \\ u \in X. \end{cases} \quad (2.3)$$

where $f(s) = |s|^{p-1}s$ and $g(s) = |s|^{q-1}s$.

Let $k \geq 1$ be a fixed integer. We are looking for solutions to (2.1) of the form

$$u(x) = \sum_{i=1}^k a_i P_s U_{\lambda_i, \xi_i/\varepsilon^\alpha}(x) + \phi_\varepsilon(x) \quad (2.4)$$
where \( a_i = +1 \) or \( a_i = -1 \), \( \lambda_1, \ldots, \lambda_k \) are positive parameters and \( \xi_1, \ldots, \xi_k \) are different points in \( \Omega \),

\[
P_\varepsilon U_{\lambda_i, \xi_i, \varepsilon \alpha}(x) = i^*_\varepsilon(U^p_{\lambda_i, \xi_i, \varepsilon \alpha})(x) \quad x \in \Omega_\varepsilon,
\]

and the function \( \phi_\varepsilon \) is a lower order term.

Let us fix some notation. Let \( y_i := \xi_i / \varepsilon^\alpha \) and let us denote for \( i = 1, \ldots, k \)

\[
\psi^0_i(x) = \frac{\partial U_{\lambda_i, y_i}}{\partial \lambda_i} = C_N N - 2 \lambda_i \frac{N-2}{2} \frac{|x - y_i|^2 - \lambda^2}{(\lambda^2 + |x - y_i|^2)^{N/2}}, \quad x \in \mathbb{R}^N,
\]

and for \( j = 1, \ldots, N \)

\[
\psi^j_i(x) = \frac{\partial U_{\lambda_i, y_i}}{\partial y^j_i} = -C_N (N-2) \lambda_i \frac{N-2}{2} \frac{x^j_i - y^j_i}{(\lambda^2 + |x - y_i|^2)^{N/2}}, \quad x \in \mathbb{R}^N.
\]

The space spanned by \( \psi^j_i, j = 0, 1, \ldots, N \) is the set of the solutions of the linearized problem (see [6])

\[
-\Delta \psi = pU^{p-1}_{\lambda_i, y_i} \psi, \quad \text{in } \mathbb{R}^N.
\]

Moreover let

\[
P_\varepsilon \psi^j_i(x) = i^*_\varepsilon(U^p_{\lambda_i, \xi_i, \varepsilon \alpha} \psi^j_i)(x) \quad x \in \Omega_\varepsilon. \tag{2.5}
\]

Let \( \lambda := (\lambda_1, \ldots, \lambda_k) \) and \( \xi := (\xi_1, \ldots, \xi_k) \). We consider the subspace of \( X \) given by

\[
K_{\varepsilon, \lambda, \xi} = \text{span} \left\{ P_\varepsilon \psi^j_i \mid j = 0, 1, \ldots, N, \; i = 1, \ldots, k \right\}
\]

and its complementary space

\[
K_{\varepsilon, \lambda, \xi}^\perp = \left\{ \phi \in X \mid \langle \phi, P_\varepsilon \psi^j_i \rangle = 0, \; j = 0, 1, \ldots, N, \; i = 1, \ldots, k \right\}.
\]

Moreover let us define the operator

\[
\Pi_{\varepsilon, \lambda, \xi}(u) = \sum_{i=1}^k \sum_{j=0}^N \langle u, P_\varepsilon \psi^j_i \rangle P_\varepsilon \psi^j_i \quad \text{and} \quad \Pi_{\varepsilon, \lambda, \xi}(u) = u - \Pi_{\varepsilon, \lambda, \xi}(u).
\]

**Definition 2.4** Let \( \delta \in (0, 1) \). Set

\[
\mathcal{O}_\delta = \{ (\lambda, \xi) \in \mathbb{R}^k_+ \times \partial \Omega \mid \lambda_i \in (\delta, \delta^{-1}), \; \text{dist}(\xi_i, \partial \Omega) \geq \delta, \; |\xi_i - \xi_h| \geq \delta, \; i \neq h \}.
\]

It is easy to prove the following result (see Lemma 2.6 in [18]).
Lemma 2.5 For any $\delta \in (0,1)$ there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $(\lambda, \xi) \in O_\delta$ and $\varepsilon \in (0, \varepsilon_0)$ it holds
\[
\|\Pi_{\varepsilon, \lambda, \xi}(u)\| \leq c\|u\| \quad \forall \ u \in X.
\]

Our approach to solve problem (2.3) will be to find, for some $\delta$, a pair $(\lambda, \xi) \in O_\delta$ and a function $\phi \in K_{\varepsilon, \lambda, \xi}$ such that
\[
\Pi_{\varepsilon, \lambda, \xi}(V_{\varepsilon, \lambda, \xi} + \phi - i_\varepsilon^*[f(V_{\varepsilon, \lambda, \xi} + \phi) + \varepsilon^q w(\varepsilon^q y)g(V_{\varepsilon, \lambda, \xi} + \phi)]) = 0 \quad (2.6)
\]
and
\[
\Pi_{\varepsilon, \lambda, \xi}(V_{\varepsilon, \lambda, \xi} + \phi - i_\varepsilon^*[f(V_{\varepsilon, \lambda, \xi} + \phi) + \varepsilon^q w(\varepsilon^q y)g(V_{\varepsilon, \lambda, \xi} + \phi)]) = 0. \quad (2.7)
\]
Here
\[
V_{\varepsilon, \lambda, \xi} := \sum_{i=1}^k a_i P_{\varepsilon} U_i \quad \text{and} \quad U_i := U_{\lambda_i, \xi_i} / \varepsilon^\alpha.
\]

Now we find, for $(\lambda, \xi) \in O_\delta$ and for small $\varepsilon$, a function $\phi \in K_{\varepsilon, \lambda, \xi}$ such that (2.6) is fulfilled.

Let us define the linear operator $L_{\varepsilon, \lambda, \xi} : K_{\varepsilon, \lambda, \xi} \rightarrow K_{\varepsilon, \lambda, \xi}$ by
\[
L_{\varepsilon, \lambda, \xi}(\phi) = \phi - \Pi_{\varepsilon, \lambda, \xi}^\perp i_\varepsilon^*[f'(V_{\varepsilon, \lambda, \xi})\phi].
\]

Arguing as in Proposition 3.1 in [18] and Lemma 1.7 in [19] we can prove the following result.

Proposition 2.6 For any $\delta > 0$ there exists $\varepsilon_1 > 0$ and a constant $C > 0$ such that, for every $(\lambda, \xi) \in O_\delta$ and for every $\varepsilon \in (0, \varepsilon_1)$, the operator $L_{\varepsilon, \lambda, \xi}$ is invertible and it holds
\[
\|L_{\varepsilon, \lambda, \xi}\phi\| \geq C\|\phi\| \quad \forall \phi \in K_{\varepsilon, \lambda, \xi}. \quad (2.8)
\]

By using Proposition 2.6 we can solve Equation (2.6).

Proposition 2.7 Let $\alpha = \frac{2}{N-6+q(N-2)}$. For any $\delta > 0$ there exist $\mu, \varepsilon_0 > 0$ such that for every $(\lambda, \xi) \in O_\delta$ and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi_{\varepsilon, \lambda, \xi} \in K_{\varepsilon, \lambda, \xi}$ such that
\[
\Pi_{\varepsilon, \lambda, \xi}^\perp(V_{\varepsilon, \lambda, \xi} + \phi - i_\varepsilon^*[f(V_{\varepsilon, \lambda, \xi} + \phi) + \varepsilon^q w(\varepsilon^q y)g(V_{\varepsilon, \lambda, \xi} + \phi)]) = 0. \quad (2.9)
\]
Moreover

\[
\|\phi\| \leq \begin{cases} 
\mu \varepsilon^{2q+2} & \text{if } N \geq 7, \ q \geq 1 \\
\mu \varepsilon^{4q} \log \varepsilon & \text{if } N = 6, \ q \geq 1 \\
\mu \varepsilon^{\alpha(N-2)} & \text{if } N = 4, 5, q > \frac{p}{2} \\
\mu \varepsilon^{\alpha(N-2)} \log \varepsilon & \text{if } N = 4, 5, q = \frac{p}{2} \\
\mu \varepsilon^{1+\alpha q} N^{-2} & \text{if } N = 4, 5, 1 \leq q < \frac{p}{2}. 
\end{cases}
\] (2.10)

Proof
First of all we point out that \(\phi\) solves equation (2.9) if and only if \(\phi\) is a fixed point of the operator \(T_{\varepsilon,\lambda,\xi} : K_{\varepsilon,\lambda,\xi} \rightarrow K_{\varepsilon,\lambda,\xi}\) defined by

\[
T_{\varepsilon,\lambda,\xi}(\phi) = L_{\varepsilon,\lambda,\xi}^{-1} \Pi_{\varepsilon,\lambda,\xi} L_{\varepsilon,\lambda,\xi}^* \left[ f(V_{\varepsilon,\lambda,\xi} + \phi) - \sum_{i=1}^{k} a_i f(U_i) - f'(V_{\varepsilon,\lambda,\xi})\phi \right] + \varepsilon^q w(\varepsilon^\alpha y) g(V_{\varepsilon,\lambda,\xi} + \phi). 
\]

The claim will follow by showing that \(T_{\varepsilon,\lambda,\xi}\) is a contraction map. From Remark 2.2, Lemma 2.3, Lemma 2.5 and Proposition 2.8 we get the estimate

\[
\|T_{\varepsilon,\lambda,\xi}\phi\| \\
\leq c \|f(V_{\varepsilon,\lambda,\xi} + \phi) - f(V_{\varepsilon,\lambda,\xi}) - f'(V_{\varepsilon,\lambda,\xi})\phi\| \frac{2N}{N+2} \\
+ c \|f(V_{\varepsilon,\lambda,\xi} + \phi) - f(V_{\varepsilon,\lambda,\xi}) - f'(V_{\varepsilon,\lambda,\xi})\phi\| \frac{N}{N+2} \\
+ c \|f(V_{\varepsilon,\lambda,\xi}) - \sum_{i=1}^{k} a_i f(U_i)\| \frac{2N}{N+2} \\
+ c \|f(V_{\varepsilon,\lambda,\xi}) - \sum_{i=1}^{k} a_i f(U_i)\| \frac{N}{N+2} \\
+ c \|w(\varepsilon^\alpha y) g(V_{\varepsilon,\lambda,\xi} + \phi)\| \frac{2N}{N+2} + \|w(\varepsilon^\alpha y) g(V_{\varepsilon,\lambda,\xi} + \phi)\| \frac{N}{N+2}. 
\] (2.11)

First of all we have

\[
\|f(V_{\varepsilon,\lambda,\xi} + \phi) - f(V_{\varepsilon,\lambda,\xi}) - f'(V_{\varepsilon,\lambda,\xi})\phi\| \frac{2N}{N+2} \\
\leq c \|\phi\|^{\min(2,p)} 
\] (2.12)
and by interpolation (since $\frac{2N}{N-2} \leq \min\{2, p\} \frac{N_s}{N+2+2s} \leq s$)

$$
\| f(V_{\varepsilon, \lambda, \xi} + \phi) - f(V_{\varepsilon, \lambda, \xi}) - f'(V_{\varepsilon, \lambda, \xi})\phi \| \leq c\||\phi\|^{\min\{2, p\}}.
$$

(2.13)

If $1 \leq q \leq p$ we get

$$
\| w(\varepsilon^a y)g(V_{\varepsilon, \lambda, \xi} + \phi) \| \leq c\|w\|_{\infty} \left( \sum_i \| (P_{\varepsilon U_i})^q \|_{\frac{2N}{N+2}} + \|\phi\|_{\frac{2N}{N+2}} \right)

\leq c \left( \sum_i \| (P_{\varepsilon U_{i, \xi + \varepsilon^a}})^q \|_{\frac{2N}{N+2}} + \varepsilon^{-\frac{q}{2}}(N+2)-(N-2)\|\phi\|_{\frac{2N}{N+2}} \right).
$$

(2.14)

Moreover if $q > p$ from Lemma 5.3, we get by interpolation (since $\frac{2N}{N-2} \leq \frac{N_s}{N+2s} \leq s$)

$$
\| w(\varepsilon^a y)g(V_{\varepsilon, \lambda, \xi} + \phi) \| \leq c\|w\|_{\infty} \left( \sum_i \| (P_{\varepsilon U_i})^q \|_{\frac{Ns}{N+2s}} + \|\phi\|_{\frac{Ns}{N+2s}} \right)

\leq c \left( \sum_i \| (P_{\varepsilon U_{i, \xi + \varepsilon^a}})^q \|_{\frac{Ns}{N+2s}} + \varepsilon^{-\frac{q}{2}}(N+2)-(N-2)\|\phi\|_{\frac{2N}{N+2}} \right).
$$

(2.15)

and also (since $\frac{2N}{N-2} \leq q \frac{Ns}{N+2s} \leq s$)

$$
\| w(\varepsilon^a y)g(V_{\varepsilon, \lambda, \xi} + \phi) \| \leq c\|w\|_{\infty} \left( \sum_i \| (P_{\varepsilon U_i})^q \|_{\frac{Ns}{N+2s}} + \|\phi\|_{\frac{Ns}{N+2s}} \right)

\leq c \left( \sum_i \| (P_{\varepsilon U_{i, \xi + \varepsilon^a}})^q \|_{\frac{Ns}{N+2s}} + \varepsilon^{-\frac{q}{2}}(N+2)-(N-2)\|\phi\|_{\frac{2N}{N+2}} \right).
$$

(2.16)

Let $1 \leq q \leq p$. By (2.11), (2.12), (2.14), (5.4) and (5.2) we deduce that if $\|\phi\| \leq \mu \varepsilon^\gamma$ in (2.10) then

$$
\| T_{\varepsilon, \lambda, \xi} \phi \| \leq c \left( \|\phi\|^{\min\{2, p\}} + \chi_1(\varepsilon) + \chi_2(\varepsilon)\varepsilon^\eta + \varepsilon\|\phi\|^{q} \right)

\leq \mu \varepsilon^\gamma.
$$

(2.17)

Let $q > p$. By (2.11), (2.12), (2.13), (2.15), (2.16), (5.2) and (5.3) we deduce that if $\|\phi\| \leq \mu \varepsilon^\gamma$ as in (2.10) then

$$
\| T_{\varepsilon, \lambda, \xi} \phi \| \leq c \left( \|\phi\|^{\min\{2, p\}} + \chi_1(\varepsilon) + \varepsilon^{\alpha(N-2)} + \varepsilon^\eta + \varepsilon\|\phi\|^{q} \right)

\leq \mu \varepsilon^\gamma.
$$

(2.18)
Arguing in a similar way (see also Proposition 3.2 in [18]) we prove that for some $L \in (0, 1)$ and for any $\|\phi_1\|, \|\phi_2\| \leq \mu \varepsilon^7$ as in (2.10) it holds

$$\|T_{\varepsilon, \lambda, \xi}(\phi_1) - T_{\varepsilon, \lambda, \xi}(\phi_2)\| \leq L \|\phi_1 - \phi_2\|.$$ 

That concludes our proof.

3 The reduced problem

In this section we will find $$(\lambda, \xi)$$ such that also Equation (2.7) is verified, namely for $i = 1, \ldots, k$ and $j = 0, 1, \ldots, N$

\[
0 = \left( V_{\varepsilon, \lambda, \xi} + \phi - i_\varepsilon^*[f(V_{\varepsilon, \lambda, \xi} + \phi)], P_{\varepsilon} \psi_i^j \right) \\
- \left( i_\varepsilon^* [\varepsilon^n w(\varepsilon^\alpha y) g(V_{\varepsilon, \lambda, \xi} + \phi)], P_{\varepsilon} \psi_i^j \right). 
\]

(3.1)

As in Proposition 2.7 we assume $\alpha = \frac{2N - 6 + q(N - 2)}{2}$. Arguing as in Proposition 2.1 of [19], we can prove the following result, that analyse the term in (3.1) independent on $q$.

Lemma 3.1 It holds if $i = 1, \ldots, k$ and $j = 1, \ldots, N$

\[
\left( V_{\varepsilon, \lambda, \xi} + \phi - i_\varepsilon^*[f(V_{\varepsilon, \lambda, \xi} + \phi)], P_{\varepsilon} \psi_i^j \right) \\
= -A^2 \left[ a_i \lambda_i^{N-2} \partial H(\xi_i, \xi_i) - \sum_{l \neq i} a_l \lambda_i^{\frac{N-2}{2}} \lambda_l^{\frac{N+2}{2}} \partial G(\xi_i, \xi_l) \right] \varepsilon^{\alpha(N-1)} \\
+ O(\|\phi\|^2) + \varepsilon^{\frac{N}{2}} O(\|\phi\|)
\]

and

\[
\left( V_{\varepsilon, \lambda, \xi} + \phi - i_\varepsilon^*[f(V_{\varepsilon, \lambda, \xi} + \phi)], P_{\varepsilon} \psi_0^j \right) \\
= \frac{N - 2}{2} A^2 \left[ a_i \lambda_i^{N-3} H(\xi_i, \xi_i) - \sum_{l \neq i} a_l \lambda_i^{\frac{N-4}{2}} \lambda_l^{\frac{N+2}{2}} G(\xi_i, \xi_l) \right] \varepsilon^{\alpha(N-1)} \\
+ O(\|\phi\|^2) + \varepsilon^{\frac{N-2}{2}} O(\|\phi\|)
\]

as $\varepsilon$ goes to zero, uniformly with respect to $(\lambda, \xi) \in \mathcal{O}_\delta$. Here

\[
A = \int_{\mathbb{R}^N} U_{1,0}^P(z) dz. 
\]

(3.2)
Arguing as in Lemma 4.2 of [18], we can estimate the term in (3.1) which depends on \( q \). At this aim we point out that it is necessary to assume \( q \neq p \).

**Lemma 3.2** Let \( q \neq p \). It holds for \( i = 1, \ldots, k \) and \( j = 1, \ldots, N \)

\[
\left( i^*_\varepsilon [w (\varepsilon^\alpha y) g (V_{\varepsilon, \lambda, \xi} + \phi), P_\varepsilon \psi^0_i] \right)
\]

\[
= - \frac{1}{q + 1} \int_{\mathbb{R}^N} U_{1,0}^{q+1} (z) dz.
\] (3.3)

Let us give the expansion of (3.1).

**Proposition 3.3** Let \( q \neq p \). It holds for \( i = 1, \ldots, k \) and \( j = 1, \ldots, N \)

\[
\left( V_{\varepsilon, \lambda, \xi} + \phi - i^*_\varepsilon [f (V_{\varepsilon, \lambda, \xi} + \phi), P_\varepsilon \psi^0_i] \right)
\]

\[
= \frac{A^2}{2} \sum_{i=1}^{k} \frac{H (\xi_i, \xi_i)}{\lambda_i} \lambda_i^{-2} - \frac{\sum_{i,l=1, i \neq l}^{k} a_i a_l G (\xi_i, \xi_l)}{\lambda_i \lambda_l} \lambda_i^{-2} \lambda_l^{-2}
\]

\[
- B \sum_{i=1}^{k} w (\xi_i) \lambda_i^{-2 (\nu - q)}.
\] (3.6)
The claim follows by Lemma 3.1 and Lemma 3.2, taking in account also that \( \alpha(N - 1) = \eta + \alpha \), since \( \alpha = \frac{2}{N-6+q(N-2)} \).

We can prove the following sufficient condition.

**Theorem 3.4** Let \((\lambda_0, \xi_0)\) be a stable critical point of the function \( \Psi_k \) (see Definition 5.4). Then for \( \varepsilon \) small enough there exists a family of solutions

\[
u_k = \sum_{i=1}^{N} a_i P_k \frac{U}{\lambda_i, \xi_i, \varepsilon} + \phi_{\lambda_i, \xi_i, \varepsilon}
\]

of problem (2.1). Moreover \( \lim_{\varepsilon \to 0} \lambda_i \varepsilon = \lambda_i^* \) and \( \lim_{\varepsilon \to 0} \xi_i \varepsilon = \xi_i^* \) where \((\lambda^*, \xi^*)\) is a critical point of \( \Psi_k \) with \( \Psi_k(\lambda_0, \xi_0) = \Psi_k(\lambda^*, \xi^*) \).

**Proof** By Proposition 3.3 and (5.7) of Definition 5.4 we get that for \( \varepsilon \) small enough there exists \((\lambda_\varepsilon, \xi_\varepsilon)\) in a neighbourhood of \((\lambda_0, \xi_0)\) such that

\[
\nabla \Psi_k(\lambda_\varepsilon, \xi_\varepsilon) + o(1) = 0.
\]

(3.7)

up to a subsequence we can assume that \( \lambda_\varepsilon \to \lambda^* \) and \( \xi_\varepsilon \to \xi^* \). By (3.7) we deduce that \((\lambda^*, \xi^*)\) is a critical point of the function \( \Psi_k \) and by (5.6) of Definition 5.4 it follows that \( \Psi_k(\lambda_0, \xi_0) = \Psi_k(\lambda^*, \xi^*) \).

It is clear that if \((\lambda_0, \xi_0)\) is an isolated critical point of \( \Psi_k \) then \((\lambda_0, \xi_0) = (\lambda^*, \xi^*) \).

**Proof of Theorem 1.2.**

Let \( q > p \) and \( \max_{x \in \Omega} w(x) < 0 \). We will prove that problem (2.1) for \( \varepsilon \) small enough has a family of solutions \( u_\varepsilon = P_k \frac{U}{\lambda_1, \xi_1, \varepsilon} \).

We assume that in (2.4) \( k = 2, a_1 = +1 \) and \( a_2 = -1 \). In this case the function \( \Psi_2 : \mathcal{M} \to \mathbb{R} \), introduced in (3.6), reduces to

\[
\Psi_2(\lambda, \xi) = \frac{A^2}{2} \left[ H(\xi_1, \xi_2) \lambda_1^{N-2} + H(\xi_2, \xi_1) \lambda_2^{N-2} + 2G(\xi_1, \xi_2) \lambda_1^{N-2} \lambda_2^{N-2} \right] - B \left[ w(\xi_1) \lambda_1^{N-2} + w(\xi_2) \lambda_2^{N-2} \right],
\]

(3.8)

where

\[
\mathcal{M} = \{ (\lambda_1, \lambda_2, \xi_1, \xi_2) \mid \lambda_1 > 0, \lambda_2 > 0, \xi_1 \in \Omega, \xi_2 \in \Omega, \xi_1 \neq \xi_2 \}. \quad (3.9)
\]

The function \( \Psi_2 \) is bounded from below on \( \mathcal{M} \). Moreover it is easy to see that

\[
\Psi_2(\lambda, \xi) \to +\infty \quad \text{as} \quad (\lambda, \xi) \to \partial \mathcal{M}.
\]
Therefore there exists a critical point of $\Psi_2$, i.e. the global minimum point, which is stable according to Definition 5.4. Finally the claim follows by Theorem 3.4.

**Proof of Theorem 1.3.**

Let $\max\{1, p-1\} \leq q < p$ and $\min w(x) > 0$. We will prove that problem (2.1) for $\epsilon$ small enough has a family of solutions $u_\epsilon = P_\epsilon U_{\lambda_1 \epsilon, \xi_1 \epsilon / \epsilon^n} - P_\epsilon U_{\lambda_2 \epsilon, \xi_2 \epsilon / \epsilon^n} + \phi_{\lambda_\epsilon, \xi_\epsilon / \epsilon^n}$.

As in the proof of Theorem 1.2, we need to consider function $\psi_2$ introduced in (3.8) constrained on the set $M$ given in (3.9).

First of all the function $\Psi_2$ is bounded from below on $M$. In fact if

$$T := \min_{\lambda > 0, \xi \in \Omega} \left[ \frac{A^2}{2} H(\xi, \xi) \lambda^{N-2} - B w(\xi) \lambda^{N-2} (p-q) \right],$$

then $\psi_2(\lambda, \xi) \geq 2T$. Moreover if $(\lambda_n, \xi_n)$ is a minimizing sequence, it is easy to see that $\lambda_n$ is bounded. Up to a subsequence, we can assume that $\lambda_n \to \lambda_0$ and $\xi_n \to \xi_0$. We want to prove that

$$(\lambda_0, \xi_0) \notin \partial M. \quad (3.10)$$

If (3.10) holds then there exists a critical point of $\Psi_2$, i.e. the global minimum point, which is stable according to Definition 5.4 and the claim will follow by Theorem 3.4.

In order to prove (3.10) we will prove that

$$\min_{\partial M} \psi_2 > \inf_{M} \psi_2. \quad (3.11)$$

Firstly it is easy to check that

$$\min_{\partial M} \psi_2 = T. \quad (3.12)$$

Secondly we exhibite a point $(\lambda, \xi) \in M$ so that

$$\psi_2(\lambda, \xi) < T. \quad (3.13)$$

In fact if $\lambda_1 > 0$ and $\xi_1 \in \Omega$ are such that

$$T = \frac{A^2}{2} H(\xi_1, \xi) \lambda_1^{N-2} - B w(\xi_1) \lambda_1^{N-2} (p-q)$$
we can evaluate
\[ \psi_2(\lambda_1, \lambda_2, \xi_1, \xi_2) = T + \frac{A^2}{2} \left[ H(\xi_2, \xi_2)\lambda_2^{N-2} + 2G(\xi_1, \xi_2)\lambda_1^{N-2} \right] \]
\[ - Bw(\xi_2)\lambda_2^{N-2} \]
and we can choose \( \lambda_2 \) small enough (and if \( p - q = 1 \) we have also to choose \( \xi_2 \) close enough to the boundary) so that
\[ \lambda_2^{N-2} \left( p - q \right) \left\{ \frac{A^2}{2} \left[ H(\xi_2, \xi_2)\lambda_2^{N-2} + 2G(\xi_1, \xi_2)\lambda_1^{N-2} \right] - Bw(\xi_2) \right\} < 0. \]

Finally (3.11) follows by (3.12) and (3.13).

Let \( \Omega_1, \ldots, \Omega_k \), be \( k \) smooth bounded domains such that \( \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \) when \( i \neq j \). Without loss of generality, we can assume that
\[ \Omega_i \subset \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \mid a_i \leq x_1 \leq b_i \} \text{ with } b_{i-1} < a_i. \]
For any \( \delta > 0 \) let
\[ C_\delta = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} \mid x_1 \in (a_1, b_k), \ |x'| \leq \delta \}. \]
Set \( \Omega_0 := \Omega_1 \cup \ldots \cup \Omega_k \) and let \( \Omega_\delta \) be a smooth connected domain such that \( \Omega_0 \subset \Omega_\delta \subset \Omega_0 \cup C_\delta \).

**Proof of Theorem 1.4.** Let \( \Omega_\delta \) defined as above. We will prove that if \( \delta \) is small enough the function \( \Psi_{\Omega_\delta} : \mathbb{R}^k_+ \times \Omega_\delta \longrightarrow \mathbb{R} \) defined by (see (3.6)
\[ \Psi_{\Omega_\delta}(\lambda, \xi) = \frac{A^2}{2} \sum_{i=1}^k \left[ H_{\Omega_\delta}(\xi_i, \xi_i)\lambda_i^{N-2} - \sum_{l=1}^k \frac{a_l}{a_i} G_{\Omega_\delta}(\xi_i, \xi_l)\lambda_l^{N-2} \lambda_i^{N-2} \right] \]
\[ - B \sum_{i=1}^k w(\xi_i)\lambda_i^{N-2} \]
has a stable critical point whenever we choose \( a_1, \ldots, a_k \in \{-1, +1\} \). The claim will follow by Theorem 3.4.

By Lemma 3.5 it follows that as \( \delta \rightarrow 0 \) it holds \( \Psi_{\Omega_\delta} \rightarrow \Psi_{\Omega_0} \) \( C^1 \)-uniformly on compact sets of \( \mathbb{R}_+^k \times \{(x_1, \ldots, x_k) \in \Omega_0 \times \ldots \times \Omega_0 \mid x_i \neq x_j \} \). Therefore, since \( \Psi_{\Omega_0} \) has a strict minimum point in the connected component \( \mathbb{R}_+^k \times \Omega_1 \times \ldots \times \Omega_k \), it follows that also \( \Psi_{\Omega_\delta} \) has a strict minimum point provided \( \delta \) is small enough.

Arguing as in [19], it is easy to prove the following result
Lemma 3.5 It holds

\[
\lim_{\delta \to 0} H_{\Omega_{\delta}}(x, x) = H_{\Omega_0}(x, x) \ \text{uniformly on compact sets of } \Omega_0
\]

and

\[
\lim_{\delta \to 0} G_{\Omega_{\delta}}(x, y) = G_{\Omega_0}(x, y) \ \text{uniformly on compact sets of } \Omega_0 \times \Omega_0 \setminus \{x = y\}.
\]

4 The symmetric case

Let us prove Theorem 1.5.

We consider the supercritical problem (i.e. \( q > p \))

\[
\begin{cases}
-\Delta u = |u|^{p-1}u - \varepsilon \lambda |u|^{q-1}u & \text{in } B \\
u = 0 & \text{on } B,
\end{cases}
\]

and the subcritical problem (i.e. \( q < p \))

\[
\begin{cases}
-\Delta u = |u|^{p-1}u + \varepsilon \lambda |u|^{q-1}u & \text{in } B \\
u = 0 & \text{on } B,
\end{cases}
\]

where \( B \) is a ball in \( \mathbb{R}^N \). Let \( h \geq 1 \) be a fixed integer and let \( k = 2h \). Set

\[
\xi_j = \left( \cos \frac{2\pi}{k}i, \sin \frac{2\pi}{k}i, 0 \right) \quad \text{for any } i = 1, \ldots, k. \quad (4.3)
\]

Without loss of generality we can assume that \( B = \{ x \in \mathbb{R}^N \mid |x| < R \} \)

for some \( R > 1 \) so that \( \xi_1, \ldots, \xi_k \in B \).

Step 1 We reduce the problem to a finite dimensional one.

We are looking for solutions to problem (4.1) or to problem (4.2) of the form

\[
u_{\varepsilon}(x) = \sum_{i=1}^{k} (-1)^{i+1} P_B U_{\lambda, y_i}(x) + v_{\varepsilon}(x),
\]

where the concentration parameter are all equal to

\[
\lambda = d \varepsilon^{\frac{N-2}{2(N-2)}}, \quad \text{with } d > 0,
\]

the concentration points are for \( i = 1, \ldots, k \)

\[
y_i = \rho \xi_i = \left( \rho \cos \frac{2\pi}{k}i, \rho \sin \frac{2\pi}{k}i, 0 \right) \quad \text{with } \rho \in [0, R],
\]
the rest term $v_\varepsilon$ is symmetric with respect to each of the variables $x_2, \ldots, x_N$
and is symmetric with respect to each line $\{t\xi_i \mid t \in \mathbb{R}\}$ for $i = 1, \ldots, k$.

Using results obtained in previous sections and taking into account the
symmetry of the domain, we reduce the problem of finding solutions to (4.1)
or to (4.2) to that of finding stable critical points (according to Definition
5.4) of the function $\psi_k : [0, R] \times \mathbb{R}^+ \to \mathbb{R}$ defined by (see (3.2) and (3.3))

$$\psi_k(\rho, d) = k \left[ \frac{1}{2} A^2 d^{N-2} \gamma_k(\rho) + Bd^{\frac{N-2}{2}(p-q)} \right] \quad \text{if } q > p, \quad (4.4)$$
or
$$\psi_k(\rho, d) = k \left[ \frac{1}{2} A^2 d^{N-2} \gamma_k(\rho) - Bd^{\frac{N-2}{2}(p-q)} \right] \quad \text{if } q < p, \quad (4.5)$$

where

$$\gamma_k(\rho) = H(\rho\xi_1, \rho\xi_1) - \sum_{i=2}^{k} (-1)^{i+1} G(\rho\xi_1, \rho\xi_i). \quad (4.6)$$

It is well known that the Green’s function $G$ of the ball $B$ is

$$G(x, y) = \alpha_N \left( \frac{1}{|x-y|^{N-2}} - \frac{1}{|x|^{N-2} - \frac{R}{|x|}} \right).$$

Moreover the Robin’s function $H$ of the ball $B$ is

$$H(x, x) = \alpha_N \frac{R^{N-2}}{(R^2 - |x|^2)^{N-2}}.$$

Step 2 \textbf{We prove that }$\psi_k$ \textbf{has a stable critical point.}

Firstly we prove that

$$\gamma_k \text{ has a minimum point } \rho_0 > 0 \text{ with } \gamma_k(\rho_0) > 0. \quad (4.7)$$

In order to simplify the computation we can assume $\alpha_N = 1$. By the choice
made in (4.3) we have

$$H(\rho\xi_1, \rho\xi_1) = \frac{R^{N-2}}{(R^2 - \rho^2)^{N-2}}$$

and for $i = 2, \ldots, k$

$$G(\rho\xi_1, \rho\xi_i) = \frac{1}{2 \frac{N-2}{2} \rho^{N-2}(1 - a_i)^{\frac{N-2}{2}}} - \frac{R^{N-2}}{(\rho^4 + R^4 - 2R^2 \rho^2 a_i)^{\frac{N-2}{2}}},$$
where (since \( k = 2h \))
\[
a_i = (\xi_1, \xi_i) = \cos \frac{\pi}{R} (i - 1).
\]
Moreover taking in account that \( a_i = a_{i+1} \) for \( i = 2, \ldots, h \) the function \( \gamma_k : [0, R) \to \mathbb{R} \) defined in (4.5) reduces to
\[
\gamma_k(\rho) = \frac{R^{N-2}}{(R^2 - \rho^2)^{N-2}} - (-1)^{h+2} \left[ \frac{1}{2^{N-2} \rho^{N-2}} - \frac{R^{N-2}}{(R^2 + \rho^2)^{N-2}} \right]
- 2 \sum_{i=2}^{h} (-1)^{i+1} \left[ \frac{1}{2^{N-2} \rho^{N-2}(1 - a_i)^{N-2}} - \frac{R^{N-2}}{(\rho^4 + R^4 - 2R^2 \rho^2 a_i)^{N-2}} \right]
= \frac{1}{R^{N-2}} \left[ \frac{1}{(1 - (\rho/R)^2)^{N-2}} + \chi(\rho/R) \right],
\]
where the function \( \chi : [0, 1) \to \mathbb{R} \) is defined by
\[
\chi(t) = \frac{a}{t^{N-2}} + \frac{(-1)^h}{(1 + t^4)^{N-2}} - 2 \sum_{i=2}^{h} (-1)^i \frac{1}{(1 + t^4 - 2t^2 a_i)^{N-2}},
\]
with
\[
a := -\frac{(-1)^h}{2^{N-2}} + \frac{2}{2^{N-2}} \sum_{i=2}^{h} (-1)^i \frac{1}{(1 - a_i)^{N-2}}.
\]
First of all by definition of \( a_i \) we deduce that
\[
a_i > a_{i+1} \quad \text{for} \quad i = 2, \ldots, h.
\]
Then it is easy to see that the constant \( a \) introduced in (4.10) satisfies
\[
a > 0.
\]
Therefore by (4.8) and (4.9) it follows that
\[
\lim_{\rho \to 0} \gamma_k(\rho) = +\infty.
\]
On the other hand it holds also that
\[
\lim_{\rho \to R} \gamma_k(\rho) = +\infty.
\]
Then by (4.13) and (4.14) it follows that the function \( \gamma_k \) has a minimum point \( \rho_0 \in (0, R) \). We have to prove that \( \gamma_k(\rho_0) > 0 \). At this aim we want to show that
\[
\chi(t) \geq 0 \quad \forall \ t \in (0, 1];
\]
By (4.8), (4.9) and (4.15) the claim will follow. By (4.12) we deduce that

\[ \lim_{t \to 0} \chi(t) = +\infty. \quad (4.16) \]

Let us compute

\[ \chi'(t) = (N - 2) \left[ -\frac{a}{t^N} + \frac{(-1)^h 2t}{(1 + t^2)^N} - 4t \sum_{i=2}^{h} \frac{(-1)^i (t^2 - a_i)}{(1 + t^4 - 2t^2 a_i)^{N/2}} \right]. \quad (4.17) \]

If \( \chi(t) = 0 \) by (4.9) we deduce that

\[ \frac{a}{t^N} = -(-1)^h \frac{2}{(1 + t^2)^N} + 2 \sum_{i=2}^{h} \frac{(-1)^i}{(1 + t^4 - 2t^2 a_i)^{N/2}} \]

and by (4.17) we get

\[ \chi'(t) = (N - 2) \frac{1 - t^4}{t} \left[ \frac{(-1)^h}{(1 + t^2)^N} - 2 \sum_{i=2}^{h} \frac{(-1)^i}{(1 + t^4 - 2t^2 a_i)^{N/2}} \right] < 0, \]

because of monotonicity condition (4.11). Therefore we have proved that

\[ \chi(t) = 0 \implies \chi'(t) < 0. \quad (4.18) \]

On the other hand it holds

\[ \chi(1) = \chi'(1) = 0, \quad \chi''(1) = \frac{N - 2}{2N - 2} \left[ \frac{(-1)^h}{(1 + t^2)^N} - 2 \sum_{i=2}^{h} \frac{(-1)^i}{(1 + t^4 - 2t^2 a_i)^{N/2}} \right] > 0. \quad (4.19) \]

Then (4.15) follows by (4.16), (4.18) and (4.19).

Finally, since \( \gamma_k(\rho_0) > 0 \), there exists \( d_0 > 0 \) such that

\[ \psi_k(\rho_0, d_0) = \min_{d \in \mathbb{R}^+} \psi_k(\rho_0, d). \quad (4.20) \]

By (4.7) and (4.20) we deduce that \( (\rho_0, d_0) \) is a minimum point of \( \psi_k \), which is stable critical point of the function \( \psi_k \) according to Definition 5.4.

5 Appendix

Set for \( y \in \mathbb{R}^N \) and \( \lambda > 0 \)

\[ PU_{\lambda, y}(x) = i^*_{\Omega} (U^\lambda_{\lambda, y})(x), \quad x \in \Omega \]
and
\[ P_{\varepsilon}U_{\lambda,y}(z) = \iota_{\Omega_{\varepsilon}}^*(U_{\lambda,y}^p)(z), \quad z \in \Omega_{\varepsilon} \] (see Definition 2.1).

In particular it holds
\[ PU_{\varepsilon,\lambda,\varepsilon,\alpha}(x) = \varepsilon^{-N/2} P_{\varepsilon}U_{\lambda,y}(x/\varepsilon^\alpha), \quad x \in \Omega. \tag{5.1} \]

Firstly we recall the following result (see [21]).

**Lemma 5.1** We have
\[ PU_{\varepsilon,\lambda,\varepsilon,\alpha}(x) = U_{\varepsilon,\lambda,\varepsilon,\alpha}(x) - A(\varepsilon\alpha\lambda)^{N/2} H(x,\xi) + o\left(\varepsilon^{\alpha(N-2)}\right), \quad x \in \Omega \]
and (see (3.2))
\[ PU_{\varepsilon,\lambda,\varepsilon,\alpha}(x) = A(\varepsilon\alpha\lambda)^{N/2} G(x,\xi) + o\left(\varepsilon^{\alpha(N-2)}\right), \quad x \in \Omega \]
as \( \varepsilon \longrightarrow 0 \) uniformly on compact sets of \( \Omega \setminus \{\xi\} \), where \( A \) is defined in (3.2).

By Lemma 5.1 and by (5.1) it follows that (see Appendix A in [19]):

**Lemma 5.2** For any \( \delta > 0 \) and for any \( \varepsilon_0 > 0 \) there exists \( c > 0 \) such that for any \( (\lambda,\xi) \in O_\delta \) and for any \( \varepsilon \in (0,\varepsilon_0) \) it holds
\[ \chi_1(\varepsilon) := \left\| \sum_{i=1}^k a_i P_{\varepsilon} U_{\lambda_i,\xi_i}/\varepsilon^\alpha - \sum_{i=1}^k a_i f(U_{\lambda_i,\xi_i}/\varepsilon^\alpha) \right\|_{2^{N/2}} \leq \begin{cases} \varepsilon \alpha^{N/2} & \text{if } N \geq 7, \\ \varepsilon^{4\alpha} |\log \varepsilon| & \text{if } N = 6, \\ \varepsilon^{\alpha(N-2)} & \text{if } N = 3, 4, 5. \end{cases} \tag{5.2} \]
Moreover
\[ \left\| f\left( \sum_{i=1}^k a_i P_{\varepsilon} U_{\lambda_i,\xi_i}/\varepsilon^\alpha \right) - \sum_{i=1}^k a_i f\left(U_{\lambda_i,\xi_i}/\varepsilon^\alpha\right) \right\|_{2^{N/2}} \leq \varepsilon^{\alpha(N-2)}. \tag{5.3} \]

A direct computation proves the following result.
Lemma 5.3 Let $N \geq 3$. For any $\delta > 0$ and for any $\varepsilon_0 > 0$ there exists $c > 0$ such that for any $(\lambda, \xi) \in \mathcal{O}_\delta$ and for any $\varepsilon \in (0, \varepsilon_0)$ it holds for any $q \geq 1$

$$
\chi_2(\varepsilon) := \sum_{i=1}^{k} \left\| (P_\varepsilon U_{\lambda_i, \xi_i/\varepsilon^n})^q \right\|^{\frac{2N}{N+2}} \leq \begin{cases} 
c & \text{if } q > \frac{p}{2}, \\
|\log \varepsilon| & \text{if } q = \frac{p}{2}, \\
c\varepsilon^{-\frac{N+2}{2}}(1-\frac{2q}{p}) & \text{if } q < \frac{p}{2}.
\end{cases} 
$$

Moreover if $q > \frac{2}{N-2}$ then

$$\sum_{i=1}^{k} \left\| (P_\varepsilon U_{\lambda_i, \xi_i/\varepsilon^n})^q \right\|^{\frac{N}{N-2}} \leq c. \quad (5.5)$$

Let us introduce the definition of stable critical point we use in this paper.

Definition 5.4 Let $g : D \rightarrow \mathbb{R}$ be a $C^1$–function, where $D \subset \mathbb{R}^N$ is an open set. We say that $\xi_0$ is a stable critical point of $g$ if $\nabla g(\xi_0) = 0$ and there exists a neighbourhood $V \subset D$ of $\xi_0$ such that $\nabla g(\xi) \neq 0$ for any $\xi \in \partial V$,

$$\nabla g(\xi) = 0, \quad \xi \in V \implies g(\xi) = g(\xi_0) \quad (5.6)$$

and

$$\deg (\nabla g, V, 0) \neq 0, \quad (5.7)$$

where $\deg$ denotes the Brouwer degree.

References


